Formulation of the horizontal pressure gradient force (PGF) in generalized coordinates.

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In a hydrostatic fluid ($\partial\phi/\partial s = -\alpha\partial p/\partial s$), the layer mass-weighted horizontal pressure gradient force (PGF) satisfies

$$\frac{\partial p}{\partial s}\left[\alpha\nabla_s p + \nabla_s\phi\right] = \nabla_s\left(\frac{\partial p}{\partial s}\alpha p\right) + \frac{\partial}{\partial s}\left(p\nabla_s\phi\right). \tag{1}$$

The 3-dimensional gradient form shown on the right indicates that net accelerations of a fluid system can only be caused by boundary forces. This has well-known implications for vortex spinup/spindown: Given that the curl of the r.h.s. of (1) reduces to

$$\frac{\partial}{\partial s}\left(\nabla_s \times p\nabla_s\phi\right),$$

we can state that interface pressure torques governing vortex spinup/spindown in individual s coordinate layers have the form $(\nabla_s \times p\nabla_s\phi)$.

It is important to preserve the above aspects when numerically solving the fluid dynamics equations. The task before us, therefore, is to find a finite-difference expression for the PGF term $[\alpha\nabla_s p + \nabla_s\phi]$ in the horizontal momentum equation that can, after multiplication by the layer thickness, be transformed by finite difference operations into an analog of the right-hand side of (1).

We start by writing the x component of the last term in (1) in the simplest possible, and thus plausibe, form $\delta_s(\overline{p}^x\delta_x\phi)$. Finite-difference product differentiation rules allow this to be expanded as follows:

$$\begin{aligned}
\delta_s(\overline{p}^x\delta_x\phi) &= (\delta_s\overline{p}^x)\delta_x\overline{\phi}^s + \overline{p}^{xs}\delta_x\delta_s\phi \\
&= (\delta_s\overline{p}^x)\delta_x\overline{\phi}^s - \overline{p}^{xs}\delta_x(\alpha\delta_s p) \\
&= (\delta_s\overline{p}^x)\delta_x\overline{\phi}^s - \delta_x\left(\overline{p}^s\alpha\delta_s p\right) + \overline{\alpha\delta_s p}^x\delta_x\overline{p}^s
\end{aligned}$$

A finite-difference equation analogous to (1) is now obtained by rearranging terms and adding an analogous expression for the y component:

$$\overline{\alpha\delta_s p}^x \delta_x \overline{p}^s + (\delta_s \overline{p}^x)\delta_x \overline{\phi}^s = \delta_x\left(\overline{p}^s \alpha\delta_s p\right) + \delta_s(\overline{p}^x \delta_x \phi)$$

$$\overline{\alpha\delta_s p}^y \delta_y \overline{p}^s + (\delta_s \overline{p}^y)\delta_y \overline{\phi}^s = \delta_y\left(\overline{p}^s \alpha\delta_s p\right) + \delta_s(\overline{p}^y \delta_y \phi)$$

The two equations above state that, in order to preserve the conservation properties expressed by (1), the finite-difference PGF must be evaluated in the form

$$\alpha\nabla_s p + \nabla_s \phi = \begin{pmatrix} \frac{\overline{\alpha\delta_s p}^x}{\overline{\delta_s p}^x}\delta_x \overline{p}^s + \delta_x \overline{\phi}^s \\ \\ \frac{\overline{\alpha\delta_s p}^y}{\overline{\delta_s p}^y}\delta_y \overline{p}^s + \delta_y \overline{\phi}^s \end{pmatrix} \qquad (2)$$

The salient result of our analysis is that writing the undifferentiated factor α in the PGF formula as simply $\overline{\alpha}^x$ or $\overline{\alpha}^y$ can lead to spurious momentum and vorticity generation. To avoid this pitfall, α must appear in the PGF formula in layer thickness-weighted form.

For use in isopycnal or quasi-isopycnals models, it is convenient to express the PGF in terms of the Montgomery potential $M = \phi + p\alpha$. The proper finite-difference analog of M in a staggered vertical grid (p and ϕ carried on layer interfaces, α carried within a layer) is

$$M = \overline{\phi}^s + \alpha\overline{p}^s.$$

The identity $\delta_s(\alpha\overline{p}^s) = \overline{\alpha\delta_s p}^s + p\delta_s\alpha$ allows us to expand the s derivative of M into

$$\delta_s M = p\delta_s\alpha + \overline{\delta_s\phi + \alpha\delta_s p}^s.$$

from which we can extract finite-difference analogs of the two common forms of the hydrostatic equation, $\partial\phi/\partial p = -\alpha$ and $\partial M/\partial\alpha = p$:

$$\partial\phi/\partial p = -\alpha \qquad \longrightarrow \qquad \delta_s\phi = -\alpha\delta_s p$$

$$\partial M/\partial\alpha = p \qquad \longrightarrow \qquad \delta_s M = p\delta_s\alpha.$$

We now write the x component of (2) as

$$\frac{\overline{\alpha\delta_s p}^x}{\overline{\delta_s p}^x}\delta_x \overline{p}^s + \delta_x \overline{\phi}^s = \delta_x M + \left[\frac{\overline{\alpha\delta_s p}^x}{\overline{\delta_s p}^x}\delta_x \overline{p}^s - \delta_x\left(\alpha\overline{p}^s\right)\right]. \qquad (3)$$

Making use of the relation

$$\overline{AB}^x - \overline{A}^x\overline{B}^x = \frac{1}{4}(\delta'_x A)(\delta'_x B)$$

where δ'_x represents the difference between two neighboring grid points, i.e., $\delta'_x = \Delta x\ \delta_x$, the term in square brackets in (3) can be expanded into

$$\begin{aligned}\frac{1}{\overline{\delta_s p}^x}\left(\overline{\alpha\delta_s p}^x - \overline{\alpha}^x\overline{\delta_s p}^x\right)\delta_x\overline{p}^s - \overline{p}^{sx}\delta_x\alpha &= \frac{1}{4\overline{\delta_s p}^x}(\delta'_x\delta_s p)(\delta'_x\alpha)\delta_x\overline{p}^s - \overline{p}^{sx}\delta_x\alpha \\ &= \frac{1}{4\overline{\delta_s p}^x}\left[(\delta'_x\delta_s p)(\delta'_x\overline{p}^s) - 4\overline{p}^{sx}\delta_s\overline{p}^x\right]\delta_x\alpha.\end{aligned}$$

The above expression involves a total of four p points, located one grid distance Δx apart on two consecutive s surfaces. Substantial simplification of this expression is possible by labeling the four points as

$$\begin{aligned}p_1 &= p\left(x - \frac{\Delta x}{2}, s - \frac{\Delta s}{2}\right) & p_2 &= p\left(x + \frac{\Delta x}{2}, s - \frac{\Delta s}{2}\right) \\ p_3 &= p\left(x - \frac{\Delta x}{2}, s + \frac{\Delta s}{2}\right) & p_4 &= p\left(x + \frac{\Delta x}{2}, s + \frac{\Delta s}{2}\right).\end{aligned}$$

With a modest amount of arithmetic, it can now be shown that the term in square brackets in (3) reduces to

$$\frac{p_1 p_2 - p_3 p_4}{(p_4 - p_2) + (p_3 - p_1)}\delta_x\alpha.$$

This term, which in combination with the term $\delta_x M$ gives the PGF in x direction, is the sought-after finite-difference analog of $-p\partial\alpha/\partial x$ in

$$\alpha\nabla_s p + \nabla_s\phi = \nabla_s M - p\nabla_s\alpha.$$

The finite difference expression for the PGF in y direction is analogous.